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Motion of a spin-$\frac{1}{2}$ particle in shape invariant scalar and magnetic fields

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Abstract. We study the motion of a spin-$\frac{1}{2}$ particle in a scalar as well as a magnetic field within the framework of supersymmetric quantum mechanics. We also introduce the concept of shape invariant scalar and magnetic fields and it is shown that the problem admits exact analytical solutions when such fields are considered.

1. Introduction

The concept of supersymmetry in quantum mechanical models was first introduced by Nicolai [1]. A few years later Witten introduced supersymmetric quantum mechanics (SUSYQM) [2] as a laboratory to examine supersymmetry breaking in quantum field theoretical models. Subsequently SUSYQM has proved to be interesting in its own right and has been studied by many authors from different points of view [3, 4].

Over the years it has been shown [3, 4] that SUSYQM plays an important role in obtaining exact solutions of quantum mechanical problems. In fact, all solvable problems of quantum mechanics are either supersymmetric (SUSY) or can be made so. Now, among the various exactly solvable potentials there is a certain class of potentials which are characterized by a property known as shape invariance [5]. Potentials which are shape invariant satisfy certain conditions and it has been shown [3–5] that solutions of the Schrödinger equation with any shape invariant potential can be obtained in a trivial manner without solving the differential equation. In fact, shape invariance is a sufficient condition for exact solvability.

In this paper our aim is to use the formalism of SUSYQM to study the one-dimensional motion of a spin-$\frac{1}{2}$ particle in the presence of a scalar potential as well as another function which can be viewed as a 'magnetic field'. Physically, this can be interpreted as the motion of an electron along a quantum wire placed in a magnetic field [6, 7]. It may be noted that SUSYQM has previously been used to study the motion of a particle in a magnetic field [8–10]. However, in the present case the problem is similar in nature to a coupled channel problem which in the context of SUSYQM was first considered by Amado *et al* [11]. Subsequently a number of similar problems have also been studied [12–15]. To solve this problem we shall introduce a definition of shape invariance which will require not only the scalar potential but also the magnetic field to satisfy certain conditions. Using this shape invariance property we shall then obtain exact solutions of the problem of a spin-$\frac{1}{2}$ particle moving in a scalar potential

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and a magnetic field. The organization of the paper is as follows: in section 2 we describe the construction of the Hamiltonian describing the motion of a spin-$\frac{1}{2}$ particle in a scalar potential and a magnetic field; in section 3 we introduce the shape invariance conditions and use them to obtain algebraically exact solutions; finally section 4 is devoted to a conclusion.

2. Supersymmetric approach to the motion of a spin-$\frac{1}{2}$ particle on the real line

In Witten's model of SUSYQM the Hamiltonian consists of two factorized Schrödinger operators

$$H_{\mp}(z;\gamma) = A^{\pm}(z;\gamma)A^{\mp}(z;\gamma) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + W^2(z;\gamma) \mp W'(z;\gamma) \tag{1}$$

where γ denotes a set of parameters and the operators $A^{+}(z;\gamma)$ and $A^{-}(z;\gamma)$ are given by

$$A^{\pm}(z;\gamma) = \mp\frac{\mathrm{d}}{\mathrm{d}z} + W(z;\gamma) \tag{2}$$

where the function $W(z;\gamma)$ is called the superpotential.

The pair of Hamiltonians in (1) are called SUSY partner Hamiltonians and each of these Hamiltonians describes the motion of a spinless particle in one-dimensional potentials $V_{\pm}(z;\gamma) = W^2(z;\gamma) \pm W'(z;\gamma)$. Among the various potentials $V_{\pm}(z;\gamma)$, those which satisfy the relation

$$V_{+}(z;\gamma) = V_{-}(z;\gamma_1) + \epsilon_1 \tag{3}$$

where $\gamma_1 = f(\gamma)$ is a function of γ and ϵ_1 is a constant, are called shape invariant potentials [5]. The shape invariant potentials are always exactly solvable and their solutions can be obtained purely algebraically.

We shall now generalize Witten's model of SUSYQM in such way that each of the Hamiltonians H_{-}, H_{+} will describe the motion of a spin-$\frac{1}{2}$ particle in a magnetic field and a scalar potential. In order to do this we generalize the operators $A^{\pm}$ in the following way:

$$A^{\pm}(z;\gamma,\beta) = \mp\frac{\mathrm{d}}{\mathrm{d}z} + W(z;\gamma) + \boldsymbol{V}(z;\beta)\boldsymbol{S}. \tag{4}$$

It may be noted that here we consider motion of the particle along the z-axis and components of the spin operator $\boldsymbol{S}$ are $S_\alpha = \sigma_\alpha/2$ $(\alpha = x, y, z)$, σ_α being the Pauli matrices. Then SUSY partner Hamiltonians can be obtained as in (1) and are given by

$$H_{\pm}(z;\gamma,\beta) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_{\pm}(z;\gamma,\beta) + \boldsymbol{B}_{\pm}(z;\gamma,\beta)\boldsymbol{S} \tag{5}$$

where

$$V_{\pm}(z;\gamma,\beta) = W^2(z;\gamma) \pm W'(z;\gamma) + V^2(z;\beta)/4 \tag{6}$$

$$\boldsymbol{B}_{\pm}(z;\gamma,\beta) = 2W(z;\gamma)\boldsymbol{V}(z;\beta) \pm \boldsymbol{V}'(z;\beta). \tag{7}$$

The Hamiltonians $H_{\pm}$ in (5) describe a spin-$\frac{1}{2}$ particle moving along the z-axis in a scalar potential $V_{\pm}(z;\gamma,\beta)$ and a magnetic field $\boldsymbol{B}_{\pm}(z;\gamma,\beta)$. Let us note that, from the mathematical point of view, this generalization of SUSY is strictly equivalent to the coupled-channel generalization of [11] for two channels.

In the present case the Hamiltonian (5) can be thought to describe an electron moving along the z-axis (on which the wire is situated) and placed in a magnetic field $\boldsymbol{B}_{\pm}(x, y, z)$. The resulting spin–magnetic field interaction is given by $\boldsymbol{S}\boldsymbol{B}_{\pm}(x, y, z)$. However, since the motion of electron is essentially along the z-axis the effective spin–magnetic field interaction is given by

$\boldsymbol{SB}_{\pm}(x, y, z)|_{x=y=0} = \boldsymbol{SB}_{\pm}(z)$ where $\boldsymbol{B}_{\pm}(0, 0, z) = \boldsymbol{B}_{\pm}(z)$. Obviously div$\boldsymbol{B}_{\pm}(x, y, z) = 0$ but div$\boldsymbol{B}_{\pm}(0, 0, z)$ can be nonzero. We note that since div$\boldsymbol{B}_{\pm}(x, y, z) = 0$ at $x = 0$, $y = 0$ Maxwell's equation is not violated. We further point out that for the purpose of our paper it is not necessary to know the magnetic field $\boldsymbol{B}_{\pm}(x, y, z)$ but it is sufficient to know only the value of the magnetic field on the z-axis. For the sake of convenience we shall henceforth refer to $\boldsymbol{B}_{\pm}(z)$ as the magnetic field.

The SUSY Hamiltonian reads

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \{Q^+, Q_-\} \tag{8}$$

where the supercharges Q^+ and Q_- have the form

$$Q^+ = A^- \otimes \sigma^+ \qquad Q^- = A^+ \otimes \sigma^-. \tag{9}$$

The supercharges and SUSY Hamiltonian fulfil the well known $N = 2$ SUSY algebra

$$\{Q^+, Q^-\} = H \qquad [Q^{\pm}, H] = 0 \qquad (Q^{\pm})^2 = 0. \tag{10}$$

Note that in the present case the SUSY Hamiltonian and supercharges act on a four component wavefunction. The standard Witten model of SUSYQM can be reproduced by setting $\boldsymbol{V} = 0$.

The Hamiltonians H_+ and H_- have exactly the same energy levels (perhaps with the exception of the zero-energy state). For the zero-energy ground state the following scenarios are possible: (1) the zero-energy ground state does not exist (broken SUSY); (2) the zero-energy ground state exists for one of the Hamiltonians H_- or H_+ (exact SUSY); (3) the zero-energy ground state exists for both Hamiltonians H_- and H_+ (exact SUSY). In a previous paper [15] it was shown that this last scenario can be realized when a particle moves in a rotating magnetic field and a zero scalar potential. For the standard Witten model of SUSYQM such a situation arises when the superpotential is a periodic function [16, 17].

In this paper we shall consider the case when the zero-energy ground state exists for one of the Hamiltonians H_- or H_+, say for H_-. In this case the eigenvalues $E_n^{\pm}$ and eigenfunctions $\psi_n^{\pm}$ of the Hamiltonians $H_{\pm}$ are related by the following SUSY transformations:

$$E_{n+1}^- = E_n^+ \qquad E_0^- = 0 \tag{11}$$

$$\psi_{n+1}^- = \frac{1}{\sqrt{E_n^+}} A^+ \psi_n^+ \tag{12}$$

$$\psi_n^+ = \frac{1}{\sqrt{E_{n+1}^-}} A^- \psi_{n+1}^-. \tag{13}$$

In equations (12) and (13) the operators $A^{\pm}$ are (2×2) matrices and the wavefunctions $\psi_n^{\pm}$ are two-component wavefunctions. As a result equations (12) and (13) are matrix differential equations although in appearance they look similar to the standard SUSY transformations [3,4].

3. Shape invariant potentials and magnetic fields

In this section we shall generalize the idea of shape invariance for obtaining the exact solution of the eigenvalue problem for a spin-$\frac{1}{2}$ particle moving in both a scalar potential as well as a magnetic field. We begin with the eigenvalue problem corresponding to the Hamiltonian H_- with superpotentials $W = W(z, \gamma)$, $\boldsymbol{V}(z, \beta)$ which depend on some parameters γ and β. Since the zero-energy ground state of this Hamiltonian is annihilated by the operator $A^-(z; \gamma, \beta)$ we have

$$A^-(z; \gamma, \beta)\psi_0^-(z; \gamma, \beta) = 0. \tag{14}$$

Note that in the case of the standard Witten model of SUSY quantum mechanics this equation is a single first-order differential equation and can be easily solved, but in the present case the operator A^- is a (2×2) matrix differential operator and therefore the above equation is a set of two first-order coupled differential equations. Therefore, in general the ground state cannot be obtained in terms of the superpotentials. This is similar to the situation in SUSY quaternionic quantum mechanics [18]. We shall return to the problem of determining the ground state later. For the time being let us assume that we have a solution of equation (14).

Now let us consider the SUSY partner of $H_-(z;\gamma,\beta)$, i.e. $H_+(z;\gamma,\beta)$. If we calculate the ground state of $H_+(z;\gamma,\beta)$ we immediately find the first excited state of $H_-(z;\gamma,\beta)$ using the SUSY transformations (11)–(13). Now in order to calculate the ground state of H_+ let us rewrite it in the form

$$H_+(z;\gamma,\beta) = H_-(z;\gamma_1,\beta_1) + \epsilon_1 = A^+(z;\gamma_1,\beta_1)A^-(z;\gamma_1,\beta_1) + \epsilon_1 \qquad \epsilon_1 > 0 \tag{15}$$

where ϵ_1 is the factorization energy. The operators $A_\pm(z;\gamma_1,\beta_1)$ corresponding to $H_-(z;\gamma_1,\beta_1)$ have the same form as in (4) but with superpotentials $W = W(z,\gamma_1)$, $\boldsymbol{V} = \boldsymbol{V}(z,\beta_1)$.

We note that the wavefunction of the ground state of $H_+(z;\gamma,\beta)$ is also the wavefunction of the ground state of $H_-(z;\gamma_1,\beta_1)$, i.e., $\psi_0^+(z,\gamma,\beta) = \psi_0^-(z,\gamma_1,\beta_1)$, and it satisfies the equation

$$A^-(z;\gamma_1,\beta_1)\psi_0^-(z,\gamma_1,\beta_1) = 0. \tag{16}$$

Using the SUSY transformations we can now obtain the energy level and corresponding wavefunction of the first excited state of the Hamiltonian $H_-(z;\gamma,\beta)$:

$$E_1^- = \epsilon_1 \qquad \psi_1^- = \frac{1}{\sqrt{\epsilon_1}}A^+(z,\gamma,\beta)\psi_0^-(z,\gamma_1,\beta_1). \tag{17}$$

From (15) we can now obtain the conditions of shape invariance involving the superpotential and the magnetic field. In explicit form these conditions read

$$W^2(z;\gamma) + W'(z;\gamma) + \boldsymbol{V}^2(z;\beta)/4 = W^2(z;\gamma_1) - W'(z;\gamma_1) + \boldsymbol{V}^2(z;\beta_1)/4 + \epsilon_1 \tag{18}$$

$$2W(z;\gamma)\boldsymbol{V}(z;\beta) + \boldsymbol{V}'(z;\beta) = 2W(z;\gamma_1)\boldsymbol{V}(z;\beta_1) - \boldsymbol{V}_1'(z;\beta_1). \tag{19}$$

Equation (18) is the condition for shape invariant scalar superpotential while (19) is the equation for shape invariant magnetic field. Comparing with equation (3) we find that in the present case shape invariance conditions consist of four equations rather than a single one.

In general it is very difficult to solve these equations for superpotentials (magnetic fields) when an arbitrary magnetic field (superpotential) is prescribed. However when we consider some specific superpotential and magnetic field solutions of equations (18) and (19) can still be obtained. To this end let us choose $\boldsymbol{V}$ in the form

$$\boldsymbol{V} = g(z)\boldsymbol{a} + \beta\boldsymbol{b} \tag{20}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular unit vectors, i.e. $\boldsymbol{ab} = 0$. Then equation (18) reads

$$W^2(z;\gamma) + W'(z;\gamma) + \beta^2/4 = W^2(z;\gamma_1) - W'(z;\gamma_1) + \beta_1^2/4 + \epsilon_1 \tag{21}$$

and the vector equation (19) splits into two scalar equations

$$2W(z;\gamma)g(z) + g'(z) = 2W(z;\gamma_1)g(z) - g'(z) \tag{22}$$

$$W(z;\gamma)\beta = W(z;\gamma_1)\beta_1. \tag{23}$$

Then, from (22) we obtain

$$g(z) = \lambda e^{\int^z (W(z;\gamma_1) - W(z;\gamma))} \tag{24}$$

where λ is some constant. Here it is important to note that since $g(z)$ does not depend on the parameters γ the difference between the new and the old superpotentials $(W(z;\gamma_1) - W(z;\gamma))$ also does not depend on these parameters.

In order to satisfy equation (23) we now choose

$$W = \gamma f(z) \tag{25}$$

which leads to the following relation between the parameters:

$$\gamma\beta = \gamma_1\beta_1. \tag{26}$$

Note that only superpotentials of the form (25) ensure that the difference $(W(z;\gamma_1) - W(z;\gamma))$ is independent of the parameter γ. Thus the superpotentials (20) and (25) lead to shape invariant scalar potential and magnetic field and so the corresponding eigenvalue problem can be solved exactly.

To find the exact solutions we now continue the shape invariant construction recursively and obtain the energy levels and the corresponding wavefunctions of H_- in the following form:

$$E_n^- = \sum_{i=0}^{n} \epsilon_i \qquad \epsilon_0 = 0 \tag{27}$$

$$\psi_n^-(z;\gamma,\beta) = C_n^- A^+(z;\gamma,\beta)\ldots A^+(z;\gamma_{n-2},\beta_{n-2})A^+(z;\gamma_{n-1},\beta_{n-1})\psi_0^-(z;\gamma_n,\beta_n) \tag{28}$$

where C_n^- are normalization constants, $\psi_0^-(z;\gamma_n,\beta_n)$ is the zero-energy eigenfunction of $H_-(z;\gamma_n,\beta_n)$ which satisfies the equation $A^-(z;\gamma_n,\beta_n)\psi_0^-(z;\gamma_n,\beta_n) = 0$, $A^\pm(z;\gamma_n,\beta_n)$ and $H_-(z;\gamma_n,\beta_n)$ are of the form (4) and (5) respectively, with superpotentials $W(z;\gamma_n)$, $\boldsymbol{V}(z;\beta_n)$. In our notations $\gamma_0 = \gamma$ and $\beta_0 = \beta$.

In explicit form the equation determining $\psi_0^-(z,\gamma_n,\beta_n)$ reads

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \gamma_n f(z) + (g(z)\boldsymbol{a} + \beta_n\boldsymbol{b})\boldsymbol{S}\right)\psi_0^-(z,\gamma_n,\beta_n) = 0. \tag{29}$$

The superpotential $\gamma_n f(z)$ can be eliminated from this equation by using the following transformation:

$$\psi_0^-(z,\gamma_n,\beta_n) = \phi(z,\beta_n)\mathrm{e}^{-\int\gamma_n f(z)} \tag{30}$$

where ϕ is a two-component function which satisfies the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + (g(z)\boldsymbol{a} + \beta_n\boldsymbol{b})\boldsymbol{S}\right)\phi(z,\beta_n) = 0. \tag{31}$$

Let us now choose $\boldsymbol{a}$ parallel to the z-axis, $\boldsymbol{b}$ parallel to the x-axis. Then equation (31), which is a set of two first-order coupled differential equations, can be rewritten in the form

$$a^-\phi_1(z,\beta_n) = -\frac{\beta_n}{2}\phi_2(z,\beta_n) \tag{32}$$

$$a^+\phi_2(z,\beta_n) = \frac{\beta_n}{2}\phi_1(z,\beta_n) \tag{33}$$

where the operators $a^\pm$ are given by

$$a^\pm = \mp\frac{\mathrm{d}}{\mathrm{d}z} + g(z)/2. \tag{34}$$

The above set of first-order coupled equations can easily be transformed into second-order equations for ϕ_1 and ϕ_2 and are given by

$$a^+a^-\phi_1(z,\beta_n) = h_-\phi_1 = -\frac{\beta_n^2}{4}\phi_1(z,\beta_n) \tag{35}$$

$$a^-a^+\phi_2(z,\beta_n) = h_+\phi_2 = -\frac{\beta_n^2}{4}\phi_2(z,\beta_n). \tag{36}$$

It is interesting to note that equations (35) and (36) have the form of eigenvalue equations corresponding to $H_\pm$ of one-dimensional SUSYQM (see equation (1)) with superpotential $g(z)/2$ and $-\beta_n^2/4$ can be treated as energy which is negative in the present case. The solutions of equations (35) and (36) need not necessarily be square integrable functions. However nonsquare integrable solutions of (35) and (36) can still be used to obtain physical solutions of the original eigenvalue equation (see equation (30)).

3.1. Examples

Case 1. The simplest superpotential which we can choose is $W = \gamma z$, but in this case from (21) it follows that $\gamma_1 = \gamma$ and we find from (24) that $g = \text{const}$. As a result we obtain a magnetic field which does not change its direction. Therefore this case can be reduced to the standard Witten model of SUSYQM.

Case 2. Let us now consider the following superpotential:

$$W = \gamma \tanh(z) \qquad \gamma > 0. \tag{37}$$

Then iterating the shape invariant condition (21) n times we obtain

$$\epsilon_n = \gamma_{n-1}^2 - \gamma_n^2 + (\beta_{n-1}^2 - \beta_n^2)/4 \tag{38}$$

$$\gamma_{n-1}(\gamma_{n-1} - 1) = \gamma_n(\gamma_n + 1). \tag{39}$$

Equation (39) have two solutions with respect to γ_n, but only one of them is acceptable from the point of view of square integrability of the wavefunction and this is given by

$$\gamma_n = \gamma_{n-1} - 1 = \gamma - n. \tag{40}$$

Now iterating the relation (26) n times we obtain

$$\beta_n = \frac{\gamma_{n-1}}{\gamma_n}\beta_{n-1} = \frac{\gamma}{\gamma_n}\beta. \tag{41}$$

To determine $g(z)$ we use (24) and obtain

$$g(z) = \frac{\lambda}{\cosh(z)}. \tag{42}$$

From (42) it is seen that the function $g(z)$ indeed does not depend on the parameters appearing in the superpotential. Now using $W(z;\gamma)$ and $g(z)$ we can calculate the scalar potential and the magnetic field in which the spin-$\frac{1}{2}$ particle is moving:

$$V_\pm = \frac{\lambda^2/4 - \gamma(\gamma \mp 1)}{\cosh^2(z)} + \gamma^2 + \beta^2 \tag{43}$$

$$\boldsymbol{B}_\pm = \frac{\lambda^2}{2}(2\gamma \mp 1)\frac{\tanh(z)}{\cosh(z)}\boldsymbol{a} + 2\gamma\beta\tanh(z)\boldsymbol{b}. \tag{44}$$

Now let us study the eigenvalue equations (35) and (36). To determine whether the spectrum is finite or infinite it is now necessary to establish the maximum value of n. In the present case equation (35) reads

$$\left(-\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\lambda}{2\cosh(z)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\lambda}{2\cosh(z)}\right)\phi_1(z,\beta_n) = -\frac{\beta_n^2}{4}\phi_1(z,\beta_n). \tag{45}$$

The asymptotic behaviour of the solutions of equation (45) at $|z| \to \infty$ is given by

$$\phi_1(z,\beta_n) \underset{|z|\to\infty}{\sim} \text{const}\,\mathrm{e}^{\pm\beta_n z/2}. \tag{46}$$

Using (32) for the second component we obtain

$$\phi_2(z,\beta_n) \underset{|z|\to\infty}{\sim} \mp\text{const}\, e^{\pm\beta_n z/2}. \tag{47}$$

From (46) and (47) it is seen that the solutions are not square integrable. Now to determine the asymptotic behaviour of $\psi_0(z,\gamma_n,\beta_n)$ we use (46) and (47) in (30) and obtain

$$\psi_0^-(z,\gamma_n,\beta_n) \underset{|z|\to\infty}{\sim} \text{const}\begin{pmatrix}1\\ \mp 1\end{pmatrix}\frac{e^{\pm\beta_n z/2}}{\cosh^{\gamma_n}(z)}. \tag{48}$$

Then from the condition of square integrability of $\psi_0^-(z,\gamma_n,\beta_n)$ we obtain

$$\gamma_n > |\beta_n|/2. \tag{49}$$

Thus it follows from (49) that

$$n < \gamma - \sqrt{\gamma|\beta|/2}. \tag{50}$$

Energy levels of the Hamiltonian $H_-(z;\gamma,\beta)$ are then given by

$$E_n^- = \gamma^2 - (\gamma-n)^2 + \frac{\beta^2}{4}\left(1 - \frac{\gamma^2}{(\gamma-n)^2}\right). \tag{51}$$

From (48) it follows that there are two independent square integrable solutions of equation (29) and as a result the ground state of $H_-(z;\beta_n,\gamma_n)$ is twofold degenerate. Now, from (28) and (29) it can be shown that each energy level E_n^- is doubly degenerate.

We now proceed to determine the eigenfunctions of $H_-(z;\beta_n,\gamma_n)$ in explicit form. In order to do this we need to have the general solutions of equation (45). To solve this equation we first transform it to the equation for hypergeometric functions. Let us introduce a new variable $x = \sinh(z)$. Then equation (45) becomes

$$\left[-(1+x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} + \frac{\lambda^2}{4}\frac{1}{1+x^2} + \frac{\lambda}{2}\frac{x}{1+x^2}\right]\phi_1 = -\frac{\beta_n^2}{4}\phi_1. \tag{52}$$

We now introduce a new function f defined by the relation

$$\phi_1 = f e^{-\frac{\lambda}{2}\arctan(x)} \tag{53}$$

and use a new variable $\xi = (1-ix)/2$ to obtain from equation (45)

$$\left[(1-\xi)\xi\frac{d^2}{d\xi^2} + \left(\frac{1}{2} - i\frac{\lambda}{2} - \xi\right)\frac{d}{d\xi}\right]f = -\frac{\beta_n^2}{4}f. \tag{54}$$

This equation has two linearly independent solutions:

$$f^{(1)} = F(a,b;c;\xi) \tag{55}$$

$$f^{(2)} = \xi^{1-c}(1-\xi)^{c-a-b}F(1-a,1-b;2-c;\xi) \tag{56}$$

where $F(a,b;c;x)$ is the hypergeometric function and $a = \beta_n/2$, $b = -\beta_n/2$, $c = \frac{1}{2} - i\lambda/2$.

Then using (28) we obtain in explicit form two eigenfunctions which correspond to the same energy level given by (51). As a result we conclude once more that energy levels of $H_-(z;\beta,\gamma)$ are twofold degenerate.

Now let us analyse the reason for this double degeneracy of the energy levels of $H_-(z;\gamma,\beta)$. Note however that this double degeneracy is not related to the SUSY of the original Hamiltonian, which consists of two partner Hamiltonians $H_-(z;\gamma,\beta)$ and $H_+(z;\gamma,\beta)$. The degeneracy of $H_-(z;\gamma,\beta)$ is related to the spin degrees of freedom of the Hamiltonian and also to the existence of an additional integral of motion $T = I\sigma_y$ in the case when $W(-z) = -W(z)$, where I is a parity operator and acts according to $If(z) = f(-z)$. Also $T^2 = 1$ and thus this operator has two eigenvalues ± 1. We also note that the operator of complex conjugation R, acting according to $Rf = f^*$, commutes with $A^{\pm}(z;\gamma_i,\beta_n)$ in the case when $\boldsymbol{a}$ is parallel to the z-axis and $\boldsymbol{b}$ is parallel to the x-axis. These operators satisfy the (anti)commutation relations

$$TA^{\pm}(z;\beta_n,\gamma_n) + A^{\pm}(z;\beta_n,\gamma_n)T = 0 \tag{57}$$

$$TR + RT = 0 \tag{58}$$

$$RA^{\pm}(z;\beta_n,\gamma_n) - A^{\pm}(z;\gamma_n,\beta_n)R = 0. \tag{59}$$

Furthermore the operators R and T commute with the Hamiltonian $H_-(z;\gamma_n,\beta_n)$

$$[T, H_-(z;\gamma_n,\beta_n)] = [R, H_-(z;\gamma_n,\beta_n)] = 0. \tag{60}$$

Let us now demonstrate using the above algebra that the zero-energy level for $H_-(z;\gamma_n,\beta_n)$ is doubly degenerate. To show this let us suppose that we have at least one zero-energy ground state. As a result of the commutation relation (60) this state can be chosen also as an eigenfunction of the operator T. Thus the zero-energy ground state satisfies the equations

$$T\psi_\lambda = \lambda\psi_\lambda \tag{61}$$

$$A^-\psi_\lambda = 0 \tag{62}$$

where λ takes one of the values 1 or -1. Then from (58) and (61) it follows that $R\psi_\lambda = \psi_{-\lambda}$ Now operating R from the left on (62) and using (59) we obtain

$$A^-R\psi_\lambda = A^-\psi_{-\lambda} = 0. \tag{63}$$

Thus $\psi_{-\lambda}$ together with ψ_λ are wavefunctions of the zero-energy ground state. We can conclude that the zero-energy level of the Hamiltonian $H_-(z;\gamma_i,\beta_i)$ is doubly degenerate. Since the nth excited state of the Hamiltonian $H_-(z;\gamma,\beta)$ is related by (28) to the ground state of the Hamiltonian $H_-(z;\beta_n,\gamma_n)$ we conclude that all the energy levels of the Hamiltonian $H_-(z;\gamma,\beta)$ are doubly degenerate. Finally, we note that as a consequence of the relation (11) the zero-energy ground level of the full Hamiltonian (8) is doubly degenerate while the excited levels are fourfold degenerate.

4. Conclusions

In this paper we extend the definition of shape invariance to obtain exact solutions of the eigenvalue problem relating to the motion of a spin-$\frac{1}{2}$ particle moving in a scalar potential and a magnetic field. The shape invariance conditions are more complicated than in the standard case. This is because instead of one equation for the superpotential W in the standard case we have four equations coupling the superpotential with the components of the vector function $\boldsymbol{V}$. It has been shown that if we choose a superpotential and a magnetic field satisfying the above mentioned shape invariance condition we can obtain exact analytical solutions of the eigenvalue problem. The spectrum of the full Hamiltonian is fourfold degenerate while those of the component Hamiltonians are doubly degenerate. We have also analysed the reasons for this double degeneracy and it has been shown to be due to the existence of additional integrals of motion rather than SUSY. We feel it would be of interest to find other superpotentials and magnetic fields which are shape invariant and are thus exactly soluble.

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References

[1] Nicolai H 1976 *J. Phys. A: Math. Gen.* **9** 1497
[2] Witten E 1981 *Nucl. Phys.* B **188** 513
[3] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[4] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[5] Gendenshteyn L E 1983 *Pis. Zh. Eksp. Teor. Fiz.* **38** 299
Gendenshteyn L E 1983 *JETP Lett.* **38** 356
[6] Yanson A I, Rubio Bollinger G, van der Brom H E, Agrait N and van Ruitenbeek J M 1998 *Nature* **395** 783
[7] Imamura H, Kobayashi N, Takahashi S and Maekawa S 2000 *Phys. Rev. Lett.* **84** 1003
[8] de Crombrugghe M and Rittenberg V 1983 *Ann. Phys.* **151** 99
[9] D'Hoker E and Vinet L 1984 *Phys. Lett.* B **137** 72
[10] Haymaker R and Rau A R P 1986 *Am. J. Phys.* **54** 928
[11] Amado R D, Cannata F and Dedonder J-P 1988 *Phys. Rev.* A **38** 3797
[12] Hau L V, Golovchenko J A and Burns M M 1995 *Phys. Rev. Lett.* **75** 1426
[13] Andrianov A A, Cannata F, Ioffe M V and Nishnianidze D N 1997 *J. Phys. A: Math. Gen.* **30** 5037
[14] Lévai G and Cannata F 1999 *J. Phys. A: Math. Gen.* **32** 3947
[15] Tkachuk V M and Roy P 1999 *Phys. Lett.* A **263** 245
[16] Dunne G and Mannix J 1998 *Phys. Lett.* B **428** 115
[17] Dunne G and Feinberg J 1998 *Phys. Rev.* D **57** 1271
[18] Davies A J 1994 *Phys. Rev.* A **49** 714